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REORGANIZATION OF THE CONFIGURATION MANIFOLD AND CRITICAL SYSTEMS†

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When the dynamics of holonomic mechanical systems are described in redundant coordinates with parameter-dependent constraints, some special features arise. At certain (critical) values of the parameters, the configuration manifold of the system may have, for example, self-intersections; in the neighbourhood of such a singular submanifold one cannot introduce *Lagrangian* coordinates of the system and it becomes difficult (even impossible) to define the orbits of the "critical" system correctly by taking limits with respect to the parameter. In such cases the classical Poincaré theory of bifurcation of equilibria in conservative systems also needs some adjustment. As an example, a crank-gear mechanism is considered. © 2000 Elsevier Science Ltd. All rights reserved.

A non-linear autonomous holonomic constraint

$$f(x, y, l) = 0 \tag{1}$$

imposed on a collection of point masses, usually involves not only the coordinates x, y, but also a parameter $l(x, y, f$ may also be vector quantities). The constraint defines in x, y space a subspace of admissible positions of the mechanical system—the configuration manifold S. Variation of the parameter l defines a family of mechanical systems and a corresponding collection $S(l)$ of configuration manifolds.

There are mechanical objects which have a critical parameter value l_* such that $S(l_*)$ intersects itself at some singular point O of the x, y plane (or on some manifold P), while $S(l)$ has no such self-intersections when $l \neq l_*$. The mechanical system corresponding to this value l_* will be called critical.

The first special feature of a critical system is that one cannot introduce generalized Lagrangian coordinates on the manifold $S(l_*)$ (of course, in the neighbourhood of the point O one cannot even introduce local coordinates).

Let us assume that the critical system is initially at rest at the point O. If this is not an equilibrium position, the question arises: along which branch of the manifold $S(l_*)$ will the system subsequently move? There is no rational answer to this question. One either has to resign oneself to an obvious violation of the determinacy principle in this case or, conversely, to "save" the principle by adopting the "convection" that O is a singular equilibrium position of the system. Neither of these solutions gives rise to any trouble from a practical standpoint, since the event itself has a low probability.

The question just asked has a natural continuation. The system "enters" the position O along a certain branch of the manifold $S(l_*)$ at some non-zero velocity. Which branch of the system will it "choose" for its subsequent motion? To seek an answer to this question, it seems natural to appeal to a fundamental law of mechanics—the law of inertia. In other words, the motion of the system will continue in such a way that the velocities of its points will vary continuously! This makes the choice of a branch for subsequent motion unambiguous, and in the case of a one-dimensional manifold implies O is a point of intersection for two orbits of the critical system.

Now let us compare the orbits of the critical system with orbits of "nearby" systems, with $l_1 = l_* - \varepsilon$ and $l_2 = l_* + \varepsilon$. In accordance with the well-known ideas of bifurcation theory for the solutions of non-linear equations, as the parameter changes, the manifold $S(l_*)$ will "split" into disjoint parts S_1, S_2, in such a way that the pair $S_1(l_* - \varepsilon)$, $S_2(l_* - \varepsilon)$ will not be "similar to" the pair $S_1(l_* + \varepsilon)$, $S_2(l_* + \varepsilon)$. The configuration manifold is subject to a cardinal reorganization, which may rightly be termed "catastrophic", as the parameter l goes through the critical value l_*. This situation has several corollaries. First, constraint (1) defines two configuration manifolds when $l \neq l_*$. One of them corresponds to the given configuration of the mechanism being studied, while the other generally corresponds to another mechanism. The second system is referred to as the conjugate system. Second, $S_1(l_* - \varepsilon)$ and $S_1(l_* + \varepsilon)$, considered as $\varepsilon \to 0$, form two limit sets which are not identical with one another and consist of pieces

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of different orbits of the critical system. Consequently, the limit orbit of the mechanical system as $l \to l_* - 0(l_* + 0)$ is not an orbit of the critical (limit) system.

Now let us assume that the critical system is conservative and that its singular position O corresponds to a minimum of the potential energy relative to perturbations belonging to one orbit or another in $S(l_*)$. Consequently, O corresponds to a stable equilibrium of the critical system and its small motions about this equilibrium are small oscillations. However, the frequency of these oscillations will generally depend on the particular orbit along which the motion has been driven. In other words, a critical system may have two modes and two frequencies of small oscillations (but not their superposition!). This property should obviously be familiar to specialists in mechanical engineering.

These singular properties of mechanical systems, which depend on a parameter, contribute certain supplements to the classical Poincaré theory of the bifurcation of equilibria of conservative mechanical systems.

Supplement 1. The maximum (or minimum) of a function $\Pi(x, y)$, which exists on the manifold $S_1(l)$ for $l < l_*$ (or for $l > l_*$) cannot be continued as a function of the parameter through a singular point O if it "coalesces" at that point with a minimum (or maximum, respectively) of Π on $S_2(l)$.

Supplement 2. Two maxima of a function $\Pi(x, y)$, one on the manifold $S_1(l)$ and the other on $S_2(l)$, $l < l_*$, which "coalesce" at the point O where $l = l_*$, are conserved on both manifolds when $l > l_*$ as well. The same holds for two minima on different manifolds.

Example. Consider a crank PA ($|PA| = r$; see Fig. 1) which can rotate bout a fixed axis P, and a piston C which can perform reciprocating motion along a fixed straight line Px. The crank PA and piston C are coupled to a connecting rod AB ($|AB| = \chi$) by cylindrical hinges A, B. All the elements of the mechanism are assumed to be absolutely rigid bodies. The crank position is defined by the angle α between the segment PA and the Px axis. The piston position is defined by the coordinate x_B and the angle β between the lines AB and Px (Fig. 1).

The system is of course holonomic, since the quantities α, β and x_B satisfy the obvious geometrical relations

$$x_B = r\cos\alpha + \chi\cos\beta, \quad l\sin\alpha = \sin\beta, \quad l = r/\chi \tag{2}$$

Since x_B is uniquely defined by the angles α and β, and constraint (2) does not involve x_B, it is convenient to picture the configuration manifold S of the system in the (α, β) plane or, to be precise, in the square $(-\pi < \alpha \leq \pi, -\pi < \beta \leq \pi)$ (Fig. 2). If $0 < l < 1$, the manifold S_1 is qualitatively represented by the dash-dot curve NKN'. If $1 < l$, the manifold S_1 has the form of the dashed curve MKM'. There is a cardinal reorganization of the shape of the configuration manifold S_1 as the parameter l "passes through" the value $l_* = 1$.

Finally, in the "critical" case $l_* = 1$ the manifold $S(l_*)$ is the union of three segments: $\alpha = \beta$, $\alpha + \beta = \pi$, $\alpha + \beta = -\pi$, the last two being continuations of one another, since the system is 2π-periodic with respect to both α and β.

The choice of a generalized Lagrangian coordinate q. It is obvious that when $l < 1$ one must take $q = \alpha$; but for $l > 1$, on the contrary, $q = \beta$. It is also obvious that at $l = 1$ there is no generalized Lagrangian coordinate (although the manifold S is still one-dimensional).

Trajectories of the "critical" system. At $l = 1$ the system has two singular positions: O: $\alpha = \beta = \pi/2$ and O': $\alpha = \beta = -\pi/2$: the system may "leave" them by way of two totally different orbits $\alpha = \beta$ and $\alpha + \beta = \pi$ or $\alpha + \beta = -\pi$. There is an indeterminacy in the choice of trajectory if the motion begins from a state of rest at one of these positions which is not an equilibrium position.

Let us assume now that the motion begins from some non-singular position and that the representative point "enters" the singular point, say, O, at non-zero velocity. If the motion takes place along the segment MO, the velocity

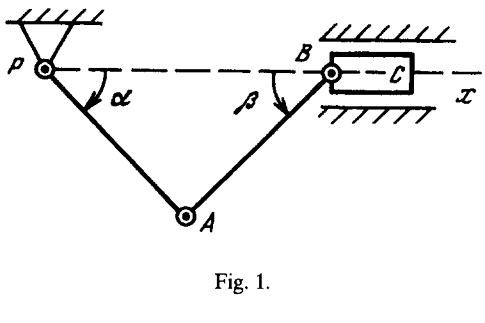

Fig. 1.

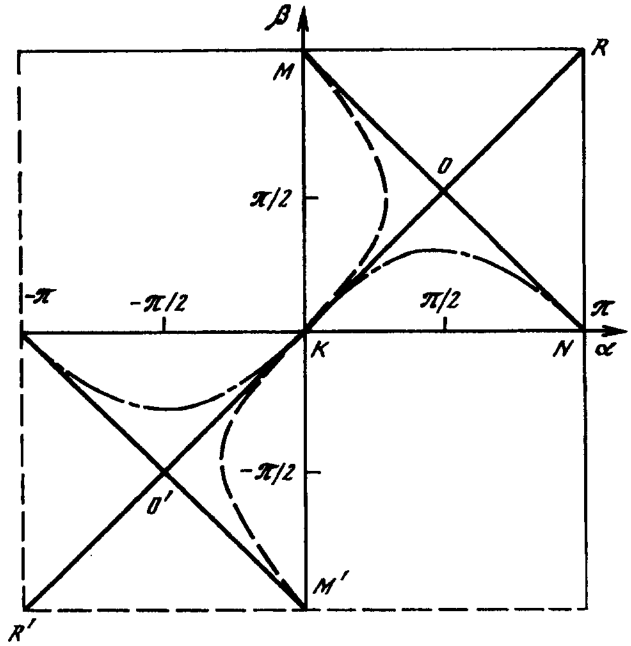

Fig. 2.

of the piston C throughout the motion is zero; but if the path is KO, the piston will move at non-zero velocity. Clearly, then, the continuation of the motion "through" the singular point O is determined by the inertia property, that is, in the first case it continues along ON, and in the second along OR. Thus, for almost all initial conditions, the segments $R'R$, $MN \cup M'N'$ must be regarded as whole trajectories of the critical system.

Limit properties. We now consider the case $l < 1$ and single out the "limit orbit" as $l \to 1 - 0$ (from the left). In view of the representation in Fig. 2, one can take the "limit" trajectory for the trajectory $N'KN$ to be the polygonal line $N'O'ON$. But if $l \to 1 + 0$ (from the right), the "limit" trajectory will have the form of the polygonal line $M'O'OM$. Thus, the "limit trajectory" need not be a trajectory of the critical (i.e., limiting) system. This means that attempts to study the trajectories of critical (limiting, degenerate) systems by letting trajectories of the non-degenerate systems tend to their limits may not yield correct results.

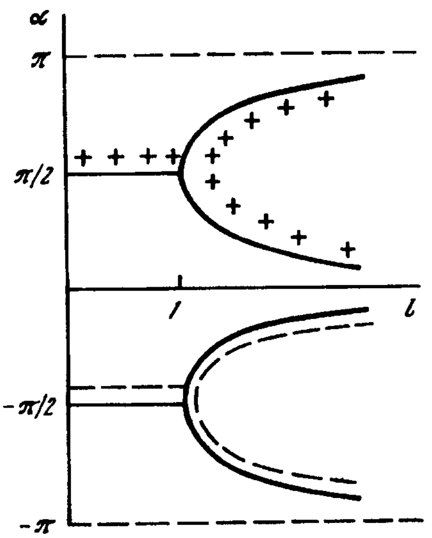

Fig. 3.

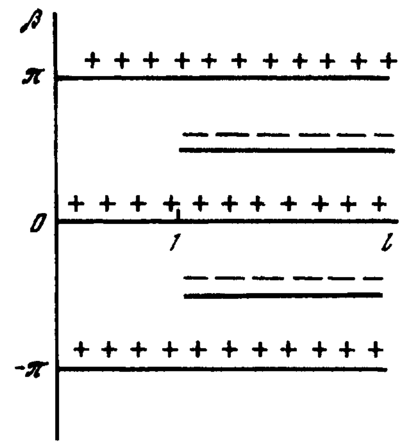

Fig. 4.

Small oscillations of the "critical system". Let us assume that the mechanism is in a uniform gravitational field with acceleration g directed perpendicular to the line Ox (downwards in Fig. 1). It is obvious that in that case the singular point $\alpha = \beta = \pi/2$ is a stable equilibrium position of the mechanism. But the frequency of small oscillations about this equilibrium obviously depends on whether the piston remains in place or not.

Assuming, for simplicity, that the rods are weightless, let us place at the hinge A a point mass m. Then the first frequency λ_1 is the frequency of a mathematical pendulum of length $r = \chi$, and the second is given by

$$\lambda_2^2 = mg/[r(m+4M)]$$

where M is the mass of the piston.

Bifurcation of equilibrium positions: In Fig. 3 we show the distribution of equilibrium positions for the mechanism in the α, l plane. It is not difficult to see that under the conditions indicated above, if $l < 1$, the equilibrium position $\alpha = \pi/2$ is stable (shown by a plus sign in Fig. 3), while $\alpha = -\pi/2$ is unstable (shown by a minus sign in Fig. 3).

If $l > 1$, the equilibrium pattern and the distribution of the stability property (Fig. 3) are clearly not in accord with the ideas of Poincaré's theory.

This was explained previously (see *Supplement* 1). On the one hand, the coordinate α can no longer play the part of a generalized *Lagrangian* coordinate when $l > 1$, since it varies within bounded limits (namely, in the strip $(-\pi/2) < \alpha < \pi/2)$). On the other hand, the second, as it were adjacent, similarly-named extremum of the potential energy (for example, in the strip $(\pi/2) < \alpha < \pi$ now belongs to the manifold S_2 of the conjugate mechanism, in which the piston C is located to the left of the point P.

To illustrate *Supplement* 2, let us assume that the uniform gravity field is acting only on the rod PA along the Px axis. In that case, if $l > 1$, constraint (2) defines not only the mechanism shown in Fig. 1 but also the "conjugate" system, for which the slide C is situated "above" the point P. The distribution of equilibria is shown in Fig. 4.

We thus see that mechanical systems whose configuration spaces may experience catastrophic change, as well as the critical systems associated with such phenomena, exhibit a number of curious properties.

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Translated by D.L.